

Partially Dissipative Systems in Locally Uniform Spaces

Alexandre N. Carvalho^{*}

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
 E-mail: andcarva@icmsc.sc.usp.br

Tomasz Dłotko[†]

Institute of Mathematics, Silesian University, 40-007 Katowice, Poland
 E-mail: tdlotko@ux2.math.us.edu.pl

In this paper we study the existence of attractors for partially dissipative systems in $\mathbb{R}^n$. For these systems, we prove the existence of global attractors with attraction properties and compactness in a slightly weaker topology than the topology of the phase spaces. We obtain abstract results extending the usual theory to encompass such two topologies attractors. These results are applied to the FitzHugh-Nagumo equations in $\mathbb{R}^n$ and to Field-Noyes Equations in $\mathbb{R}$. October, 2001 ICMC-USP

1. INTRODUCTION

The prototype problem considered in this paper is the following FitzHugh-Nagumo system in $\mathbb{R}^n$

$$\begin{cases} u_t = \Delta u - \alpha v + f(u), \\ v_t = -\delta v + \beta u + h(x), \\ u(0) = u_0, \quad v(0) = v_0. \end{cases} \quad t > 0, \quad x \in \mathbb{R}^n \quad (1)$$

where α, β, δ are positive constants and the assumptions of f and h will be made precise later.

It is known that the above system, for $n = 1$, may exhibit a relaxation wave solution (see [12]). We aim to give a result on existence of a global attractor for (1) in such a way that these relaxation wave solutions are included in it. Also, by making $h(x) \equiv 0$ one may have constant equilibria for (1) and our attractor should include these equilibria as well. There has been some efforts to obtain the existence of attractor for (1) in $\mathbb{R}^n$ (see, for example, [19]) but in this case none of the above described special solutions are in the attractor.

^{*} Partially supported by grant # 300.389/92-5 CNPq and grant # 1997/11323-0 FAPESP, Brazil

[†] Partially supported by grant # 2001/03735-4 FAPESP, Brazil and by grant KNB # 2 P03A 035 18, Poland.

This is due to the fact that the problem (1) has been set in usual Sobolev spaces (say $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$), which require that the elements in the attractor “vanishes” at infinity and therefore do not include constant functions or relaxation wave solutions.

In order to include these special solutions (equilibria and travelling waves) in the attractor we must choose sufficiently “large” spaces to work in. One attempt is to consider weighted L^p spaces as in [1, 3]. These spaces are defined in the usual way by replacing the Lebesgue measure dx with $\rho(x)dx$ and the weight ρ should be chosen such that ρ is a positive integrable $C^2(\mathbb{R}^n)$ function satisfying some additional conditions that will be specified later. The disadvantage of such spaces is that it kills the behavior of the solutions for large spatial values besides that, the usual Sobolev type embedding are not available for them. In [10] locally compact attractors are considered for damped wave equations and this idea lead many people to work in locally uniform Sobolev spaces which are the completion of the C^∞ bounded functions in $\mathbb{R}^n$ relatively to the norm $\|u\|_{W_{lu}^{m,p}(\mathbb{R}^n)} = \sup_{y \in \mathbb{R}^n} \|u\|_{W_{\rho_y}^{m,p}(\mathbb{R}^n)}$ (here $\rho_y(x) = \rho(x-y)$). In [16] the existence of global attractors for problems like the Ginzburg-Landau equation is established in the framework of locally uniform Sobolev spaces. The existence of attractors in unbounded domains has been studied in many other works, among them we cite [1, 6, 15, 4]

In [3] the theory for existence of attractors includes the possibility that the attractor is not compact in the phase space. This is also the case in [10, 16]. In [16] an abstract definition of the so called $(Z - Z_\rho)$ -attractors is given. Here Z and Z_ρ are Banach spaces with $\|\cdot\|_{Z_\rho} \leq \|\cdot\|_Z$. In this case the attractor must attract bounded subsets of Z in the norm of Z_ρ and the attractor is compact in Z_ρ . This approach differs from the theory in [7] by the fact that in the later Z and Z_ρ are the same space.

In [16] the definition of $(Z - Z_\rho)$ -attractor is abstract but the result on existence of global attractors is formulated for a specific problem. We give abstract conditions for the existence of $(Z - Z_\rho)$ -attractors and then apply it to partly dissipative parabolic partial differential equations like (1).

2. ABSTRACT RESULTS

In this section we recall the notion of an $(E - E_\rho)$ -attractor given in [16] and give conditions for the existence of such attractors. In [16] the notion of a $(E - E_\rho)$ -attractor is introduced but the existence of such attractors is left to the applications without an abstract result that provides sufficient conditions for its existence. We prove such result and show that it can be applied to many problems in unbounded domains. We follow closely the results in [7] and in [13] making the necessary changes to accommodate the lack of compactness in such problems.

A typical result on existence of global attractors states that *a point dissipative, bounded and asymptotically smooth continuous semigroup $\{T(t) : t \geq 0\}$ on a complete metric space E has a compact global attractor*. This attractor is compact, invariant, attracts bounded sets under the semigroup. We try to obtain similar conditions for the case when asymptotic compactness of the semigroup is not available in the usual phase space E but it is available in a large space E_ρ in which E is continuously embedded.

In order to present the results we introduce some notation and terminology. Let E and E_ρ be complete metric spaces and let d and d_ρ the corresponding metrics for these spaces. Assume that $E \subset E_\rho$ with continuous inclusion. Let $\{T(t) : E \rightarrow E, t \geq 0\}$ be a continuous semigroup; that is, $(t, x) \mapsto T(t)x : \mathbb{R} \times E \rightarrow E$ is a continuous map. Assume that there is a Hausdorff topology τ on E such that any bounded set of E is sequentially compact in $E^* = (E, \tau)$. Under these assumptions we may define ω -limit sets of a bounded subsets B in E which have bounded orbit $\gamma^+(B) = \{T(t)x, t \geq 0, x \in B\}$ in the following way

$$\omega(B) = \{y \in E : \text{there are sequences } t_n \rightarrow \infty, \{x_n\} \text{ in } B \text{ such that } d_\rho(y, T(t_n)x_n) = 0\}.$$

This definition uses the three spaces introduced previously since any such sequence $T(t_n)x_n$ which converges to y in E_ρ must have a subsequence that converges in E^* (to y) and therefore $y \in E$. Assume also that the closure in E^* of a bounded subset of E is again a bounded subset of E . It is important that $\omega(B)$ is a subset of E for we want it to be positively invariant under $T(t)$ and in the above setting this follows in the usual manner.

Remark 2. 1. Note that we do not claim that E is compactly embedded in E_ρ and in general that is not the case in the applications.

Next we introduce the class of semigroups that have non-empty ω -limit sets of bounded sets with bounded orbits. These are the so called $(E - E_\rho)$ -**asymptotically compact** semigroups defined as follows

DEFINITION 2.1. A continuous semigroup $\{T(t) : t \geq 0\}$ is said *asymptotically compact* if the following holds: if B is bounded subset of $E = (E, d)$ with bounded orbit $\gamma^+(B)$ in $E = (E, d)$ then, for each sequence of positive real numbers $\{t_n\}$ with $t_n \rightarrow +\infty$ and sequence $\{x_n\}$ in B we have that $\{T(t_n)x_n\}$ has a convergent subsequence in $E_\rho = (E, d_\rho)$.

This ensures that the ω -limit set of every bounded set B of $E = (E, d)$ with bounded orbit in $E = (E, d)$ is non-empty. The above definitions and notations are enough to prove the following result

PROPOSITION 2.1. *Let E , E_ρ and E^* be as above and $\{T(t) : t \geq 0\}$ be a continuous, $(E - E_\rho)$ -asymptotically compact semigroup. Suppose that B is a bounded subset of E with bounded orbit $\gamma^+(B)$ in E . Then $\omega(B)$ is non-empty, invariant, compact in E_ρ and attracts bounded subsets of E under $T(t)$ in the topology of E_ρ ; that is, $d_\rho\{T(t)B, \omega(B)\} \rightarrow 0$ as $t \rightarrow \infty$.*

Proof: Most of this proof follows in the same way as the existing proofs for similar results with only one complete metric space involved (see [7, 13]). We will go through the proof giving emphasis to the differences.

As we already mentioned $\omega(B)$ is non-empty and $\omega(B) \subset E$. Here it is important to note that E^* played an important role to ensure that $\omega(B) \subset E$. One thing we need to check and that was automatic in the case when only one space is involved is that $\omega(B)$

should be bounded in E and that comes from the fact that any bounded subset of E is such that its closure relatively to the topology of E^* also a bounded subset of E .

The proof that $\omega(B)$ is closed in E is obtained in the following way. Assume that y is in the closure in E of $\omega(B)$, then there is a sequence $\{y_n\}$ in $\omega(B)$ which converges to y in E . Choose $t_n > n$ and $x_n \in B$ such that $d_\rho(T(t_n)x_n, y_n) < \frac{1}{n}$. Then it follows from triangle inequality and from the embedding $E \hookrightarrow E_\rho$ that $d_\rho(T(t_n)x_n, y) \rightarrow 0$ as $n \rightarrow \infty$ and therefore $y \in \omega(B)$.

The invariance $T(t)\omega(B) = \omega(B)$, $t \geq 0$ is obtained exactly in the same manner as in the case when there is only one space involved. To obtain the compactness of $\omega(B)$ in E_ρ we use the invariance; that is, for each sequence $\{x_n\}$ in $\omega(B)$ there is a sequence $\{\bar{x}_n\}$ in $\omega(B)$ with $x_n = T_n(\bar{x}_n)$ and the asymptotic compactness of $\{T(t) : t \geq 0\}$ ensures the existence of a convergent subsequence in E_ρ and proves the compactness of $\omega(B)$ in E_ρ .

It remains only to prove that $\omega(B)$ attracts B under $\{T(t) : t \geq 0\}$ in E_ρ metric. This is proved as in [13] or [7] for the case when only one space is involved. $\square$

Now we are ready to state a result that gives sufficient conditions for the existence of a $(E - E_\rho)$ -attractor. Its proof is straight forward from the existing results when only one space is involved (see [13, 7]).

THEOREM 2.1. *Let $\{T(t) : t \geq 0\}$ be a continuous semigroup. Assume that there is a bounded set B_0 in E that absorbs orbits of bounded sets in E ; that is $d(T(t)B, B_0) \rightarrow 0$ as $t \rightarrow \infty$ for any bounded subset B of E . If the semigroup $\{T(t) : t \geq 0\}$ is asymptotically $(E - E_\rho)$ -compact, then it has a global attractor.*

Remark 2. 2. Here we note that if E is a reflexive Banach space we may use E^* as the space E endowed with the weak topology and any bounded subset B of E is contained in its closed convex hull $\overline{\text{co}}B$. The set $\overline{\text{co}}B$ is bounded in E and closed in E^* showing that the closure of B in E^* is a bounded subset of E . This is the typical auxiliary space E^* used in the applications.

3. APPLICATION TO FITZHUGH-NAGUMO EQUATIONS

In this section we introduce the functional framework for problem (1) in order that it can be put in the abstract setting of Section 2. We start describing the function spaces that will be used throughout. Let $\rho : \mathbb{R}^n \rightarrow (0, \infty)$ be a $C^2(\mathbb{R}^n)$ integrable weight function and denote by $L_\rho^p(\mathbb{R}^n)$, $p > 1$, the set of all functions φ in $L_{\text{loc}}^1(\mathbb{R}^n)$ such that

$$\|\varphi\|_{L_\rho^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\varphi(x)|^p \rho(x) dx \right)^{\frac{1}{p}} < \infty.$$

In a similar way one defines the spaces $W_\rho^{m,p}(\mathbb{R}^n)$.

In order to deal with energy estimates involving the use of the application of the Divergence Theorem these weights should enjoy the following additional property

$$|\nabla \rho| \leq \rho_0 \rho, \quad |\Delta \rho| \leq \text{const} \rho \quad (2)$$

where ρ can be chosen such that the constant $\rho_0 > 0$ is as small as needed. Such weights are well known in the literature (see [5]) and one example of it is $\rho_\epsilon(x) = (1 + |\epsilon x|^2)^{-n}$. As we mentioned previously these weighted spaces are not appropriate (from our point of view) to describe the dynamics of evolution equations in $\mathbb{R}^n$, instead we will work in the spaces $W_{\text{lu}}^{m,p}(\mathbb{R}^n)$ which are given as the completion of the $C^\infty(\mathbb{R}^n)$ bounded functions relatively to the norm $\|\varphi\|_{W_{\text{lu}}^{m,p}(\mathbb{R}^n)} = \sup_{y \in \mathbb{R}^n} \|\varphi\|_{W_{\rho_y}^{m,p}(\mathbb{R}^n)}$ where $\rho_y(x) = \rho(x - y)$.

Remark 3. 1. We observe that for any bounded sequence $\{u_n\}$ in $L_{\text{lu}}^2(\mathbb{R}^n)$ there is a subsequence $\{u_{n_k}\}$ and $u \in L_{\text{lu}}^2(\mathbb{R}^n)$ such that

$$u_{n_k} \xrightarrow{L_{\rho_y}^2(\mathbb{R}^n)} u, \quad \forall y \in \mathbb{R}^n. \quad (3)$$

Furthermore, if $\|u_n\|_{L_{\text{lu}}^2(\mathbb{R}^n)} \leq M$, $n \geq 1$, then $\|u\|_{L_{\text{lu}}^2(\mathbb{R}^n)} \leq M$. This is obtained in the following way: For any $y \in \mathbb{R}^n$ we have that

$$\|u_n\|_{L_{\rho_y}^2(\mathbb{R}^n)} \leq M$$

therefore there is a subsequence $\{u_{n,y}\}$ and function $u_y \in L_{\rho_y}^2(\mathbb{R}^n)$ such that

$$u_{n,y} \xrightarrow{L_{\rho_y}^2(\mathbb{R}^n)} u_y.$$

also $\|u\|_{L_{\rho_y}^2(\mathbb{R}^n)} \leq M$. By taking a countable dense subset of $\mathbb{R}^n$, using the *Cantor diagonal process* and the continuity with respect to translation of functions in $L_{\text{lu}}^2(\mathbb{R}^n)$ we obtain a subsequence with the desired properties.

It is known that $-\Delta$ defines a sectorial operator in $L_{\text{lu}}^p(\mathbb{R}^n)$ with domain $W_{\text{lu}}^{2,p}(\mathbb{R}^n)$ (see [15, 4]). Because of translation invariance property these spaces do not enjoy compact embeddings as Sobolev spaces in bounded domains, they are however compactly embedded in weighted spaces. More precisely, the following embeddings are compact

$$W_{\text{lu}}^{m,p}(\mathbb{R}^n) \hookrightarrow W_{\rho}^{j,q}(\mathbb{R}^n), \quad j - \frac{n}{q} < m - \frac{n}{p}, \quad 1 < p \leq q < \infty.$$

Beside these we have the following continuous embeddings (see [5], Chapter 4)

$$W_{\text{lu}}^{m,p}(\mathbb{R}^n) \hookrightarrow W_{\text{lu}}^{j,q}(\mathbb{R}^n), \quad j - \frac{n}{q} \leq m - \frac{n}{p}, \quad 1 < p \leq q < \infty$$

(see [16], Theorem 3.2 for a proof of the case $p = q = 2$ and $n = j = 1$).

We also quote the Nirenberg-Gagliardo type inequality

$$\|\phi\|_{L^r_{\text{lu}}(\mathbb{R}^n)} \leq C \|\phi\|_{L^q_{\text{lu}}(\mathbb{R}^n)}^{1-\theta} \|\phi\|_{W^{m,p}_{\text{lu}}(\mathbb{R}^n)}^\theta, \quad \phi \in L^q_{\text{lu}}(\mathbb{R}^n) \cap W^{m,p}_{\text{lu}}(\mathbb{R}^n) \quad (4)$$

where $\frac{1}{r} = \theta(\frac{1}{p} - \frac{m}{n}) + \frac{1-\theta}{q}$, $1 < p, q, r < \infty$, m is a positive integer and C depends on the integral of the weight function ρ only through its integral.

Let $E = H^1_{\text{lu}}(\mathbb{R}^n) \times L^2_{\text{lu}}(\mathbb{R}^n)$ and $E_\rho = H^1_\rho(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ with the usual topology given by the norm and let E^* be the space E with the convergence notion defined in (3) for bounded sequences in E . Assume that $n \geq 3$, $h \in L^2_{\text{lu}}(\mathbb{R}^n)$ and that f in (1) satisfy the following conditions

$$|f'(s)| \leq a(1 + |s|^{n^*}), \quad s \in \mathbb{R}, \quad (5)$$

where $0 \leq n^* < \frac{n+2}{n-2}$,

$$f'(s) \leq b, \quad \forall s \in \mathbb{R} \text{ for some } b > 0, \quad (6)$$

and

$$f(s)s \leq -cs^2 + d, \quad \forall s \in \mathbb{R}, \text{ for some } c, d > 0. \quad (7)$$

Under these assumptions we will show that problem (1) is globally well posed for $(u_0, v_0) \in E$ and that the semigroup generated by this problem is asymptotically $(E - E_\rho)$ -compact and bounded. This implies, from the results in Section 2, the existence of a global $(E - E_\rho)$ -attractor for (1).

Remark 3. 2. A simple example of a nonlinearity f satisfying all the above assumptions for $n = 3$ is the function $f(u) = u(1 - |u|^{p-1})$, $1 < p < 5$.

Before we proceed let us rewrite (1) in the following matrix form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u) - \alpha v \\ \beta u + h(x) \end{pmatrix}. \quad (8)$$

The above problem can be seen as an abstract semilinear parabolic problem, in $X = L^2_{\text{lu}}(\mathbb{R}^n) \times L^2_{\text{lu}}(\mathbb{R}^n)$, having the form

$$\frac{d}{dt}e = Ae + F(e), \quad (9)$$

where $e = \begin{pmatrix} u \\ v \end{pmatrix}$, $A : D(A) \subset X \rightarrow X$ is the sectorial operator given by $D(A) = H^2_{\text{lu}}(\mathbb{R}^n) \times L^2_{\text{lu}}(\mathbb{R}^n)$, $Ae = \begin{pmatrix} \Delta u \\ -\delta v \end{pmatrix}$ and $F(e) = \begin{pmatrix} f(u) - \alpha v \\ \beta u + h(x) \end{pmatrix}$. It is known that $X^{\frac{1}{2}} = E$ and using the local existence results in [11] we have that (9) is locally well posed in E . More precisely, for any initial data $e_0 \in X^{\frac{1}{2}}$ there a maximal positive time τ and a function $e \in C([0, \tau), X^{\frac{1}{2}})$ such that $e(0) = e_0$, $e \in C^1((0, \tau), X^{\frac{1}{2}})$ and (9) is satisfied. The above

smoothness properties of the solutions justify the computations in the following a priori estimates.

To obtain global existence of solutions in E we show its E -norm do not blow up in a finite time. This is accomplished through the following a priori estimates.

Firstly, we obtain a priori estimate of the solutions in X .

3.1. First a priori estimate

Multiplying the first equation in (1) by $\beta u \rho_y$, the second by $\alpha v \rho_y$, integrating over $\mathbb{R}^n$ and adding the results we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\beta \int_{\mathbb{R}^n} u^2 \rho_y(x) dx + \alpha \int_{\mathbb{R}^n} v^2 \rho_y(x) dx \right] &= \beta \int_{\mathbb{R}^n} \Delta u u \rho_y(x) dx - \alpha \delta \int_{\mathbb{R}^n} v^2 \rho_y(x) dx \\ &+ \beta \int_{\mathbb{R}^n} f(u) u \rho_y(x) dx + \alpha \int_{\mathbb{R}^n} h(x) v \rho_y(x) dx. \end{aligned} \quad (10)$$

Note that

$$\int_{\mathbb{R}^n} \Delta u u \rho_y(x) dx \leq - \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx + \frac{\rho_0}{2} \left[\int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx + \int_{\mathbb{R}^n} u^2 \rho_y(x) dx \right], \quad (11)$$

that

$$\int_{\mathbb{R}^n} f(u) u \rho_y(x) dx \leq -c \int_{\mathbb{R}^n} u^2 \rho_y(x) dx + d \int_{\mathbb{R}^n} \rho_y(x) dx \quad (12)$$

and that

$$\left| \int_{\mathbb{R}^n} h(x) v \rho_y(x) dx \right| \leq \frac{\eta}{2} \int_{\mathbb{R}^n} v^2 \rho_y(x) dx + \frac{1}{2\eta} \int_{\mathbb{R}^n} h^2(x) \rho_y(x) dx. \quad (13)$$

Choosing η and ρ_0 suitably small we have

$$\begin{aligned} \frac{d}{dt} \left[\beta \int_{\mathbb{R}^n} u^2 \rho_y(x) dx + \alpha \int_{\mathbb{R}^n} v^2 \rho_y(x) dx \right] &+ \frac{\beta}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx \\ &\leq -c\beta \int_{\mathbb{R}^n} u^2 \rho_y(x) dx - \alpha \delta \int_{\mathbb{R}^n} v^2 \rho_y(x) dx \\ &+ 2\beta d \int_{\mathbb{R}^n} \rho_y(x) dx + \frac{\alpha}{\eta} \int_{\mathbb{R}^n} h(x)^2 \rho_y(x) dx. \end{aligned} \quad (14)$$

This estimate implies that

$$\beta \int_{\mathbb{R}^n} u(t)^2 \rho_y(x) dx + \alpha \int_{\mathbb{R}^n} v(t)^2 \rho_y(x) dx \leq C(\|u(0)\|_{L_{\rho_y}^2(\mathbb{R}^n)}, \|v(0)\|_{L_{\rho_y}^2(\mathbb{R}^n)}), \quad (15)$$

for all $t \geq 0$ and as long as the local $X^{\frac{1}{2}}$ -solution of (1) exists. Here $C : \mathbb{R}^2 \rightarrow [0, \infty)$ is increasing in each argument and locally bounded. Since $(u(0), v(0)) \in X^{\frac{1}{2}}$, by taking

supremum over all y in $\mathbb{R}^n$ the estimate in (15) can be extended to the following one

$$\|(u, v)\|_{X^{\frac{1}{2}}}^2 \leq \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) C(\|u(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)}, \|v(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)}). \quad (16)$$

Moreover, returning to (14) one can see that

$$\int_t^{t+r} \int_{\mathbb{R}^n} |\nabla u(\tau)|^2 \rho_y(x) dx d\tau \leq R \quad (17)$$

where R depends only on r being independent of t .

3.2. Second a priori estimate

Now we will estimate the expression

$$\int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx,$$

which appears the $H_{\rho_y}^1(\mathbb{R}^n)$ norm of u . Multiplying the first equation in (1) by $u_t \rho_y$ and integrating over $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx = \int_{\mathbb{R}^n} \Delta u u_t \rho_y(x) dx + \int_{\mathbb{R}^n} f(u) u_t \rho_y(x) dx - \alpha \int_{\mathbb{R}^n} v u_t \rho_y(x) dx.$$

Now, integrating by parts we obtain

$$\int_{\mathbb{R}^n} \Delta u u_t \rho_y(x) dx \leq -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx + \frac{\rho_0}{2} \left[\int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx + \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx \right],$$

moreover,

$$\int_{\mathbb{R}^n} f(u) u_t \rho_y(x) dx = \frac{d}{dt} \int_{\mathbb{R}^n} F(u) \rho_y(x) dx,$$

where $F(s) = \int_0^s f(z) dz$, and using Young's inequality we find that

$$\left| \alpha \int_{\mathbb{R}^n} v u_t \rho_y(x) dx \right| \leq \frac{\eta}{2} \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx + \frac{1}{2\eta} \int_{\mathbb{R}^n} v^2 \rho_y(x) dx.$$

With the above estimates we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} |\nabla u(t)|^2 \rho_y(x) dx - \int_{\mathbb{R}^n} F(u(t)) \rho_y(x) dx \right) + \left(1 - \frac{\rho_0}{2} - \frac{\eta}{2} \right) \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx \\ \leq \frac{\rho_0}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx + \frac{1}{2\eta} \int_{\mathbb{R}^n} v^2 \rho_y(x) dx. \end{aligned} \quad (18)$$

Chose now η and ρ_0 such that; $0 < 1 - \frac{\rho_0}{2} - \frac{\eta}{2} \leq \frac{1}{2}$. Since, thanks to the assumption (7),

$$F(s) \leq -cs^2 + d \quad (19)$$

then

$$-\int_{\mathbb{R}^n} F(u)\rho_y(x)dx + d \int_{\mathbb{R}^n} \rho_y(x)dx \geq 0 \quad (20)$$

The estimates (15) and (17) allow us to use the Uniform Gronwall Lemma (see [20] page 89) to the differential inequality (18) and get that, for any $t \geq 0$ and $r > 0$ fixed,

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(t+r)|^2 \rho_y(x)dx - \int_{\mathbb{R}^n} F(u(t+r))\rho_y(x)dx + d \int_{\mathbb{R}^n} \rho_y(x)dx \leq M, \quad (21)$$

for some $M = M\left(\|u(0)\|_{L_{lu}^2(\mathbb{R}^n)}, \|v(0)\|_{L_{lu}^2(\mathbb{R}^n)}, r\right)$ with the right hand side of (21) being independent of t . From (20) and (21) we obtain a bound for $\|\nabla u(t)\|_{L_{lu}^2(\mathbb{R}^n)}$, uniform for $t \in [r, \infty)$ and for initial conditions varying in bounded subsets of X . Moreover, (18) allows us together with estimates (15), (17) and (19) to bound the second left hand side term in (18)

$$\int_0^t u_t^2(s)\rho_y(x)dxds \leq N. \quad (22)$$

where $N = N(\|u(0)\|_{H_{lu}^1(\mathbb{R}^n)}, \|v(0)\|_{L_{lu}^2(\mathbb{R}^n)}, t)$. We are ready now to give an additional a priori estimate of u .

3.3. Third a priori estimate

We only sketch it. Differentiating the first equation of (1) with respect to t , multiplying the result by $tu_t\rho_y(x)$ and integrating, we obtain

$$\int_{\mathbb{R}^n} u_{tt} t u_t \rho_y(x)dx = \int_{\mathbb{R}^n} \Delta u_t t u_t \rho_y(x)dx - \alpha \int_{\mathbb{R}^n} v_t t u_t \rho_y(x)dx + \int_{\mathbb{R}^n} f'(u) t u_t^2 \rho_y(x)dx. \quad (23)$$

Estimating each component of the above expression we obtain

$$\int_{\mathbb{R}^n} u_{tt} t u_t \rho_y(x)dx = \frac{1}{2} \int_{\mathbb{R}^n} (t u_t)^2 \rho_y(x)dx - \frac{1}{2} \int_{\mathbb{R}^n} u_t^2 \rho_y(x)dx,$$

$$\int_{\mathbb{R}^n} \Delta u_t t u_t \rho_y(x)dx \leq - \int_{\mathbb{R}^n} |\nabla u_t|^2 t \rho_y(x)dx + \frac{\rho_0}{2} \left[\int_{\mathbb{R}^n} |\nabla u_t|^2 t \rho_y(x)dx + \int_{\mathbb{R}^n} u_t^2 t \rho_y(x)dx \right]$$

and

$$\left| \alpha \int_{\mathbb{R}^n} v_t t u_t \rho_y(x)dx \right| \leq \frac{\alpha}{2} \int_{\mathbb{R}^n} u_t^2 t \rho_y(x)dx + \frac{1}{2\alpha} \int_{\mathbb{R}^n} (-\delta v + \beta u + h(x)) t \rho_y(x)dx$$

where we have used the second equation of (1) in the last estimate. Collecting the above estimates and using the growth condition (5), we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} t u_t^2 \rho_y(x) dx + \int_{\mathbb{R}^n} |\nabla u_t|^2 t \rho_y(x) dx &\leq \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx + (\rho_0 + \alpha) \int_{\mathbb{R}^n} u_t^2 t \rho_y(x) dx \\ &\quad + \frac{1}{\alpha} \int_{\mathbb{R}^n} (-\delta v + \beta u + h(x))^2 t \rho_y(x) dx. \end{aligned}$$

The estimates (15) and (22) and the usual Gronwall Lemma (see [20]) lead to the following local in time estimate:

$$\int_{\mathbb{R}^n} t u_t^2 \rho_y(x) dx \leq N_0(\|u(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)}, \|v(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)}, T),$$

valid for $t \in [0, T]$. This bound immediately implies boundedness of

$$\int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx \tag{24}$$

uniformly for $t \in [\epsilon, T]$, $0 < \epsilon < T$ and for $(u(0), v(0))$ in bounded subsets of $X^{\frac{1}{2}}$.

3.4. Fourth a priori estimate

Next we use the above estimates to bound the $L_{\text{lu}}^2(\mathbb{R}^n)$ -norm of $f(u)$. To that end we multiply the first equation in (1) by $f(u)\rho_y$ and integrate over $\mathbb{R}^n$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u_t f(u) \rho_y(x) dx &= \int_{\mathbb{R}^n} \Delta u f(u) \rho_y(x) dx - \alpha \int_{\mathbb{R}^n} v f(u) \rho_y(x) dx + \int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx \\ &= - \int_{\mathbb{R}^n} f'(u) |\nabla u|^2 \rho_y(x) dx - \int_{\mathbb{R}^n} f(u) \nabla u \nabla \rho_y(x) dx \\ &\quad - \alpha \int_{\mathbb{R}^n} v f(u) \rho_y(x) dx + \int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx. \end{aligned}$$

For the terms in the above equation we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u_t f(u) \rho_y(x) dx \right| &\leq \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx + \frac{1}{4} \int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx, \\ \left| \int_{\mathbb{R}^n} \alpha v f(u) \rho_y(x) dx \right| &\leq \int_{\mathbb{R}^n} \alpha^2 v^2 \rho_y(x) dx + \frac{1}{4} \int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx, \\ \int_{\mathbb{R}^n} f'(u) |\nabla u|^2 \rho_y(x) dx &\leq b \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx, \end{aligned}$$

$$\left| \int_{\mathbb{R}^n} f(u) \nabla u \nabla \rho_y(x) dx \right| \leq \frac{\rho_0}{2} \left[\int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx + \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx \right]$$

and choosing $\rho_0 \leq \frac{1}{2}$ these imply that

$$\int_{\mathbb{R}^n} f(u)^2 \rho_y(x) dx \leq 4 \int_{\mathbb{R}^n} u_t^2 \rho_y(x) dx + 4 \int_{\mathbb{R}^n} \alpha^2 v^2 \rho_y(x) dx + (4b+1) \int_{\mathbb{R}^n} |\nabla u|^2 \rho_y(x) dx \quad (25)$$

providing, through the previous estimates, the bound for $f(u(t))$ in $L_{\text{lu}}^2(\mathbb{R}^n)$ uniform for $t \in [\epsilon, T]$ and for $(u(0), v(0))$ in bounded subsets of $X^{\frac{1}{2}}$.

Remark 3. 3. In all the estimates for which the supremum over y must be taken one can easily see that the involved bounds do not depend on y . This is what must be done in estimates (17), (21), (22), (24) and (25).

3.5. Existence of a global attractor

The estimates obtained for u_t in (24) and for $f(u)$ in (25) will be easily, using the elliptic regularity theory, converted to an estimate for u in $H_{\text{lu}}^2(\mathbb{R}^n)$. Rewriting the first equation in (1) in the form of an “elliptic” equation

$$\Delta u = u_t + \alpha v - f(u)$$

we see that its right hand side is bounded in $L_{\text{lu}}^2(\mathbb{R}^n)$ thanks to the previous a priori estimates. From the fact that $\Delta : H_{\text{lu}}^2(\mathbb{R}^n) \subset L_{\text{lu}}^2(\mathbb{R}^n) \rightarrow L_{\text{lu}}^2(\mathbb{R}^n)$ generates an analytic semigroup (see [15, 4]) we have a $H_{\text{lu}}^2(\mathbb{R}^n)$ bound C for u valid for $t \in [\epsilon, T]$ and uniform for $(u(0), v(0))$ varying in bounded subsets of $X^{\frac{1}{2}}$;

$$\|u(t)\|_{H_{\text{lu}}^2(\mathbb{R}^n)} \leq C \left(\|u(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)}, \|v(0)\|_{L_{\text{lu}}^2(\mathbb{R}^n)} \right), \quad t \in [\epsilon, T]. \quad (26)$$

In what follows we will prove that the semigroup associated to the problem (1) is *asymptotically compact*. This is accomplished through the above a priori estimates and through the Variation of Constants Formula for (8); that is, if $T(t)(u_0, v_0) = \begin{pmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{pmatrix}$ is the semigroup associated to (8), then

$$\begin{cases} u(t, u_0, v_0) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} [f(u(s, u_0, v_0)) - \alpha v(s, u_0, v_0)] ds, \\ v(t, u_0, v_0) = e^{-\delta t} v_0 + \int_0^t e^{-\delta(t-s)} \beta u(s, u_0, v_0) ds + \frac{1-e^{-\delta t}}{\delta} h. \end{cases} \quad (27)$$

From (26) and compactness of the embedding $H_{\text{lu}}^2(\mathbb{R}^n) \hookrightarrow H_{\rho}^1(\mathbb{R}^n)$ we have that the first coordinate u of $T(t)$ is compact. For the second coordinate v of $T(t)$ it is evident from

the Variation of Constants Formula (27) that it decomposes into a part convergent to $\frac{h}{\delta}$ and an integral part which is compact in $L^2_\rho(\mathbb{R}^n)$. These assure that $T(t)$ is asymptotically compact in $E_\rho = H^1_\rho(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$.

With all the above computations and Theorem 2.1 we have proved the following result.

THEOREM 3.1. *Assuming that f satisfies (5), (6), (7) and $h \in L^2_{lu}(\mathbb{R}^n)$, the problem (1) defines a bounded dissipative $(X^{\frac{1}{2}} - E_\rho)$ -asymptotically compact semigroup on $X^{\frac{1}{2}}$ and therefore (1) has an $(X^{\frac{1}{2}} - E_\rho)$ -attractor $\mathcal{A}$.*

When $h \in H^2_{lu}(\mathbb{R}^n)$, it follows from the a priori estimate (26), from the Variation of Constant Formula (27) and from the definition of $(X^{\frac{1}{2}} - E_\rho)$ -attractor that the following holds:

COROLLARY 3.1. *The attractor for (1) is a bounded subset of $H^2_{lu}(\mathbb{R}^n) \times H^2_{lu}(\mathbb{R}^n)$ whenever $h \in H^2_{lu}(\mathbb{R}^n)$.*

4. APPLICATION TO FIELD-NOYES EQUATIONS

In this section we consider a slight generalization of the the Field-Noyes system which is a model for the Belousov-Zhabotinskii reactions in chemical kinetics (see for example [18, 8, 9, 14]). To avoid unnecessary technicalities we restrict the presentation to the case of dimension $n = 1$; that is, we consider the system

$$\begin{cases} u_t = u_{xx} + \alpha(v - uv + f(u)), \\ v_{1t} = \frac{1}{\alpha}(\gamma w - v - uv), \\ v_{2t} = \sigma(u - w), \end{cases} \quad t > 0, \quad x \in \mathbb{R} \quad (28)$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

Assuming that f is locally Lipschitz the above problem is locally well posed in $H^1_{lu}(\mathbb{R}) \times L^2_{lu}(\mathbb{R}) \times L^2_{lu}(\mathbb{R})$.

We will be interested in the solutions of the above system which start at non-negative initial data. Following [18], Chapter 14, it can be shown that for a dense subset of the cone $E = H^1_{lu}(\mathbb{R}, \mathbb{R}^+) \times L^2_{lu}(\mathbb{R}, \mathbb{R}^+) \times L^2_{lu}(\mathbb{R}, \mathbb{R}^+)$ (say $[C_b^\infty(\mathbb{R}, \mathbb{R}^+)]^3$) the solution will stay in E as long as it exists. To prove that solutions that start in E will stay in E as long as they exist we consider the following auxiliary system

$$\begin{cases} u_t = u_{xx} + \alpha(v - uv + f(u)) + \eta, \\ v_{1t} = \frac{1}{\alpha}(\gamma w - v - uv) + \eta, \\ v_{2t} = \sigma(u - w) + \eta, \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \quad \eta > 0 \quad (29)$$

with initial data in $[C_b^\infty(\mathbb{R}, \mathbb{R}^+)]^3$ and apply the reasoning used in [18]. Letting η go to zero we obtain the same conclusion for the system (28) with initial data in $[C_b^\infty(\mathbb{R}, \mathbb{R}^+)]^3$. To obtain the result for (28) in E it is enough to use continuity with respect to initial data.

As in the previous example, let $E = H_{\text{lu}}^1(\mathbb{R}, \mathbb{R}^+) \times L_{\text{lu}}^2(\mathbb{R}, \mathbb{R}^+) \times L_{\text{lu}}^2(\mathbb{R}, \mathbb{R}^+)$, $E_\rho = H_\rho^1(\mathbb{R}, \mathbb{R}^+) \times L_\rho^2(\mathbb{R}, \mathbb{R}^+) \times L_\rho^2(\mathbb{R}, \mathbb{R}^+)$ and E^* the space E with the convergence notion defined in (3) for bounded sequences in E .

Assume that f has locally bounded derivative, that it satisfies (7) with suitably large constant c and that $\sigma\alpha - \gamma^2 > 0$. Under these assumptions we prove that the semigroup generated by (28) in E is bounded dissipative, $(E - E_\rho)$ -asymptotically compact and therefore posses a $(E - E_\rho)$ -global attractor.

In what follows we sketch the a priori estimates required to obtain boundedness of the semigroup in E and $(E - E_\rho)$ -asymptotic compactness.

The first a priori estimate gives a bound for the solutions in $L_{\text{lu}}^2(\mathbb{R}, \mathbb{R}^+) \times L_{\text{lu}}^2(\mathbb{R}, \mathbb{R}^+) \times L_{\text{lu}}^2(\mathbb{R}, \mathbb{R}^+)$ and it is obtained by multiplying each equation by the corresponding variable times ρ_y , integrating and adding the results. Here we use the fact that all variables assume only nonnegative values to get rid of the terms $-u^2v$ and $-uv^2$. Using Young inequality and Divergence Theorem we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 \rho_y + v^2 \rho_y + w^2 \rho_y) dx + \frac{1}{2} \int_{\mathbb{R}} |u_x|^2 \rho_y dx &\leq -\frac{\alpha c}{2} \int_{\mathbb{R}} u^2 \rho_y dx \\ &\quad -\frac{1}{2\alpha} \int_{\mathbb{R}} v^2 \rho_y dx - \frac{\sigma}{2} \int_{\mathbb{R}} w^2 \rho_y dx + \frac{\gamma}{\alpha} \int_{\mathbb{R}} vw \rho_y dx + M_0 \end{aligned} \quad (30)$$

for some constant M_0 .

Since $\sigma\alpha - \gamma^2 > 0$ we have that $\|u\|_{L_{\text{lu}}^2(\mathbb{R})}^2 + \|v\|_{L_{\text{lu}}^2(\mathbb{R})}^2 + \|w\|_{L_{\text{lu}}^2(\mathbb{R})}^2$ remains bounded uniformly for initial data in bounded subsets of E and for $t \geq 0$. Furthermore, there is a constant $K = K(d, \rho) > 0$ such that

$$\limsup_{t \rightarrow \infty} \left(\|u\|_{L_{\text{lu}}^2(\mathbb{R})}^2 + \|v\|_{L_{\text{lu}}^2(\mathbb{R})}^2 + \|w\|_{L_{\text{lu}}^2(\mathbb{R})}^2 \right) \leq K \quad (31)$$

uniformly for initial data in bounded subsets of E . Moreover, returning to (30) one can see that

$$\int_t^{t+r} \int_{\mathbb{R}} |u_x(\tau)|^2 \rho_y(x) dx d\tau \leq R \quad (32)$$

where R depends only on r being independent of t .

For a second a priori estimate we obtain dissipation for the first coordinate u in $H_{\text{lu}}^1(\mathbb{R})$. Since this differs considerably from the estimates in the previous example we include the computations. Multiply the first equation in (28) by $u_t \rho_y$ and integrate over $\mathbb{R}$

to obtain

$$\begin{aligned}
\int_{\mathbb{R}} u_t^2 \rho_y(x) dx &= \int_{\mathbb{R}} u_{xx} u_t \rho_y(x) dx + \alpha \int_{\mathbb{R}} (w - uv + f(u)) u_t \rho_y(x) dx \\
&\leq -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_x^2 \rho_y(x) dx + \alpha \frac{d}{dt} \int_{\mathbb{R}} F(u) \rho_y(x) dx + \frac{\rho_0}{2} \int_{\mathbb{R}} u_x^2 \rho_y(x) dx \\
&\quad + \frac{\rho_0}{2} \int_{\mathbb{R}} u_t^2 \rho_y(x) dx + \alpha \int_{\mathbb{R}} v u_t \rho_y(x) dx - \alpha \int_{\mathbb{R}} v u u_t \rho_y(x) dx.
\end{aligned} \tag{33}$$

The last two term in the inequality above are estimated as follows

$$\left| \alpha \int_{\mathbb{R}} v u_t \rho_y(x) dx \right| \leq \frac{1}{4} \int_{\mathbb{R}} u_t^2 \rho_y(x) dx + k_0 \int_{\mathbb{R}} v^2 \rho_y(x) dx \tag{34}$$

for some $k_0 > 0$ and using the Nirenber-Gagliardo type inequality (4) there are constants k_1 and k_2 such that

$$\begin{aligned}
-\alpha \int_{\mathbb{R}} v u u_t \rho_y(x) dx &= -\frac{\alpha}{2} \int_{\mathbb{R}} (u^2)_t v \rho_y(x) dx \\
&= -\frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 v \rho_y(x) dx + \frac{1}{2} \int_{\mathbb{R}} u^2 (\gamma w - v - uv) \rho_y(x) dx \\
&\leq -\frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 v \rho_y(x) dx + k_1 \int_{\mathbb{R}} (u^4 \rho_y(x) + u^6 \rho_y(x) + v^2 \rho_y(x) + w^2 \rho_y(x)) dx \\
&\leq -\frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 v \rho_y(x) dx + k_2 \left(\|u\|_{L_{\text{lu}}^2(\mathbb{R})}^3 \|u\|_{H_{\text{lu}}^1(\mathbb{R})} + \|u\|_{L_{\text{lu}}^2(\mathbb{R})}^4 \|u\|_{H_{\text{lu}}^1(\mathbb{R})}^2 \right) \\
&\quad + k_1 \int_{\mathbb{R}} (v^2 \rho_y(x) + w^2 \rho_y(x)) dx
\end{aligned} \tag{35}$$

Rearranging estimate (33) and using (34), (35) we obtain

$$\begin{aligned}
\frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{R}} u_x^2 \rho_y(x) dx - \alpha \int_{\mathbb{R}} F(u) \rho_y(x) dx + \frac{\alpha}{2} \int_{\mathbb{R}} u^2 v \rho_y(x) dx \right) &+ \frac{1}{2} \int_{\mathbb{R}} u_t^2 \rho_y(x) dx \\
&\leq \frac{\rho_0}{2} \int_{\mathbb{R}} u_x^2 \rho_y(x) dx + (k_0 + k_1) \int_{\mathbb{R}} v^2 \rho_y(x) dx + k_1 \int_{\mathbb{R}} w^2 \rho_y(x) dx \\
&\quad + k_2 \left(\|u\|_{L_{\text{lu}}^2(\mathbb{R})}^3 \|u\|_{H_{\text{lu}}^1(\mathbb{R})} + \|u\|_{L_{\text{lu}}^2(\mathbb{R})}^4 \|u\|_{H_{\text{lu}}^1(\mathbb{R})}^2 \right)
\end{aligned} \tag{36}$$

Thanks to the estimates (32) and (31) we are able to apply the Uniform Gronwall Lemma (see [20] page 89) to (36) and conclude that the functional

$$\frac{1}{2} \int_{\mathbb{R}} u_x^2 \rho_y(x) dx - \alpha \int_{\mathbb{R}} F(u) \rho_y(x) dx$$

is bounded uniformly for initial data in bounded subsets of E and for $t \in [r, \infty)$, with $r > 0$ fixed. Therefore $u \in H_{\text{lu}}^1(\mathbb{R})$ with the norm bounded uniformly for initial data in bounded subsets of E and $t \in [r, \infty)$.

Also, returning to (36) we find that

$$\int_t^{t+r} \int_{\mathbb{R}} u_t^2(\tau) \rho_y(x) dx d\tau \leq k_3 \quad (37)$$

form some k_3 depending on r .

Now, using the dissipativeness condition (7) we obtain that the semigroup generated by (28) is a bounded dissipative. Next we show that this semigroup is asymptotically compact verifying that its first coordinate u is a compact operator and reasoning as in the FitzHugh-Nagumo equation for the remaining coordinates.

Our third a priori estimate is devoted to the bound of $\int_{\mathbb{R}} u_t^2 \rho_y dx$ over time intervals of the form $[\epsilon, T]$ with $0 < \epsilon < T < \infty$. This is obtained differentiating the first equation in (28) with respect to t , multiplying by $tu_t \rho_y$ and integrating the result to obtain:

$$\int_{\mathbb{R}} u_{tt} tu_t \rho_y(x) dx = \int_{\mathbb{R}} u_{txx} tu_t \rho_y(x) dx + \alpha \int_{\mathbb{R}} (v - uv + f(u))_t tu_t \rho_y(x) dx. \quad (38)$$

Note that

$$\int_{\mathbb{R}} u_{tt} tu_t \rho_y(x) dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_t^2 t \rho_y(x) dx - \frac{1}{2} \int_{\mathbb{R}} u_t^2 \rho_y(x) dx \quad (39)$$

and

$$\int_{\mathbb{R}} u_{txx} tu_t \rho_y(x) dx \leq -\frac{1}{2} \int_{\mathbb{R}} u_{tx}^2 t \rho_y(x) dx + \frac{1}{2} \int_{\mathbb{R}} u_t^2 t \rho_y(x) dx \quad (40)$$

also, since $v \geq 0$, using the second equation of (28), we find that:

$$\begin{aligned} -\alpha \int_{\mathbb{R}} (uv)_t tu_t \rho_y(x) dx &= \alpha \int_{\mathbb{R}} (u_t v + u(v)_t) tu_t \rho_y(x) dx \leq -\alpha \int_{\mathbb{R}} u u_t (v)_t t \rho_y(x) dx \\ &= -\int_{\mathbb{R}} u u_t (\gamma w - v - uv) t \rho_y(x) dx \end{aligned} \quad (41)$$

and that

$$\alpha \int_{\mathbb{R}} (v)_t tu_t \rho_y(x) dx = \int_{\mathbb{R}} (\gamma v - v - uv) tu_t \rho_y(x) dx. \quad (42)$$

Now, thanks to the local boundedness of f' , to the embedding $H_{\text{lu}}^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and to the boundedness of u in $H_{\text{lu}}^1(\mathbb{R})$ obtained in the previous a priori estimate, we have that

$$\alpha \int_{\mathbb{R}} f'(u) u_t^2 t \rho_y(x) dx \leq k_4 \int_{\mathbb{R}} u_t^2 t \rho_y(x) dx \quad (43)$$

for $t \in [r, \infty)$.

Using (39), (40), (41), (42), (43) and (38) we have

$$\frac{d}{dt} \int_{\mathbb{R}} u_t^2 t \rho_y(x) dx + \int_{\mathbb{R}} u_{tx}^2 t \rho_y(x) dx \leq \int_{\mathbb{R}} u_t^2 \rho_y(x) dx + k_5 \int_{\mathbb{R}} u_t^2 t \rho_y(x) dx + k_6$$

which gives the required uniform with respect to initial data in bounded subsets of E and to $t \in [\epsilon, T]$, $0 < r \leq \epsilon < T < \infty$ bound for u_t in $L_{\text{lu}}^2(\mathbb{R})$. Rewriting the first equation of (28) in an “elliptic” form:

$$u_{xx} = u_t - \alpha(v - uv + f(u))$$

we see that

$$u_{xx} \in L^\infty([\epsilon, T]; L_{\text{lu}}^2(\mathbb{R}))$$

and that u is bounded in $H_{\text{lu}}^2(\mathbb{R})$ uniformly for initial data in bounded subsets of E and for $t \in [\epsilon, T]$.

This last a priori estimate implies that the first coordinate u of the semigroup generated by (28) is compact in $H_\rho^1(\mathbb{R})$ and using an argument with the Variation of Constants Formula as in the previous example we concluded that this semigroup is asymptotically compact.

THEOREM 4.1. *Assuming that f has locally bounded first derivative and satisfies (7), the problem (28) defines a bounded dissipative $(E - E_\rho)$ -asymptotically compact semigroup on E and therefore (28) has an $(E - E_\rho)$ -attractor $\mathcal{A}$.*

5. FURTHER REMARKS

In this section we compare the results in this paper with the results of [19]. First note that with the assumptions in [19] and following the same a priori estimates as in Section 3 one can state that the global attractor $\mathcal{B}$ obtained in [19] is a bounded subset of $H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ and therefore a bounded subset of $H_{\text{lu}}^2(\mathbb{R}^n) \times H_{\text{lu}}^2(\mathbb{R}^n)$, as long as ρ is a bounded weight function. This together with the invariance of the attractor and with the remarks stated in the Introduction concerning travelling waves and constant stationary solutions assures that $\mathcal{B} \subsetneq \mathcal{A}$.

Also, proceeding as in Subsection 3.2 we may obtain the result in [19], Lemma 2.2 without the assumption $h'(s) \geq -C$ in it. This is the only place in that work where this condition is needed and therefore it can be dropped. In our case the weight ρ and the lack of embedding for $H_\rho^1(\mathbb{R}^n)$ into $L_\rho^p(\mathbb{R}^n)$, $p > 2$, prevents us from dropping (6).

Acknowledgement: This work has been carried out while the second author visited the Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo. He wishes to acknowledge the hospitality of the People at this institution and the support from the FAPESP.

REFERENCES

1. F. Abergel, Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains, *J. Diff. Equations* **83** (1990), 85-108.
2. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Translated and revised from the 1989 Russian original by Babin, North-Holland, Amsterdam, (1992).
3. A.V. Babin and M.I. Vishik, Attractors of partial differential equations in an unbounded domain, *Proc. Roy. Soc. Edinburgh* **116A** (1990), 221-243.
4. J. Cholewa and Tomasz Dłotko, Cauchy problems in weighted Lebesgue spaces, submitted for publication.
5. D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators*, Cambridge University Press, Cambridge, 1996.
6. J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg–Landau equation I. Compactness methods, *Physica D* **95** (1996) 191-228.
7. J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs 25, AMS, Providence, RI, 1988.
8. S. Hastings and J. Murray, The existence of oscillatory solution in the Field-Noyes model for the Belousov-Zhabotinskii reaction, *SIAM J. Appl. Math.* **28** (1975) 678-688.
9. L. Howard and N. Kopell, Plane wave solutions to reaction-diffusion systems, *Stud. Appl. Math.* **52** (1973) 291-328.
10. E. Feireisl, Bounded, locally compact global attractors for semilinear damped wave equations in $\mathbb{R}^n$, *Diff. Int. Equations* **9** (1996), 1147-1156.
11. D. Henry, *Geometric Theory of Semilinear Parabolic Problems*, Lecture Notes in Mathematics 840, Springer, Berlin, 1981.
12. L. V. Kalachev, A relaxation wave solution of the FitzHugh-Nagumo equations, *Journal of Mathematical Biology* **31** (1993) 133-147.
13. O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.
14. M. Marion, Finite-dimensional attractors associated with partly dissipative reaction-diffusion systems, *SIAM J. Math. Anal.* **20** (1989) 816-844.
15. A. Mielke, The Complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors, *Nonlinearity* **10** (1997) 199-222.
16. A. Mielke and G. Schneider, Attractors for modulation equations on unbounded domains—existence and comparison, *Nonlinearity* **8** (1995) 743-768.
17. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
18. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York (1994).
19. A. Rodriguez-Bernal and B. Wang, Attractor for partly dissipative reaction diffusion systems in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **252** (2000) 790-803.
20. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York 1988.